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$N = 8$ SUPERSINGLETON QUANTUM FIELD THEORY

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We quantize the $N = 8$ supersymmetric singleton field theory which is formulated on the boundary of the four-dimensional anti-de Sitter spacetime (AdS_4). The theory has rigid OSp(8,4) symmetry which acts as a superconformal group on the boundary of AdS_4. We show that the generators of this symmetry satisfy the full quantum OSp(8,4) algebra. The spectrum of the theory contains massless states of all higher integer and half-integer spin which fill the irreducible representations of OSp(8,4) with highest spin $s_{max} = 2, 4, 6, \dots$. Remarkably, these are in one-to-one correspondence with the generators of Vasiliev's infinite-dimensional extended higher spin superalgebra shs(8,4), suggesting that we may have stumbled onto a field-theoretic realization of this algebra. We also discuss the possibility of a connection between the $N = 8$ supersingleton theory with the eleven-dimensional supermembrane in an $AdS_4 \times S^7$ background.

1. Introduction

Supersingletons are the most fundamental representations of the super anti-de Sitter groups [1,2]. They have remarkable properties which have been discussed extensively in the literature. For example, singleton field theory can be described on the $S^p \times S^1$ boundary of AdS_{p+2} alone, as opposed to the whole of AdS_{p+2} [2–4]. Moreover, it is known that treating the singletons as preons, one can construct, on purely kinematical grounds, infinitely many massless states of all spins (massless in the anti-de Sitter sense) out of just two singletons [3–5].

The philosophy of this paper is to study the $d = 4$, $N = 8$ supersingleton theory [6,7] in its own right, although we will touch upon certain issues concerning its possible connection with the eleven-dimensional supermembrane theory [8] formulated on $AdS_4 \times S^7$ [9,10]. In a previous letter [11], we already presented the results on the spectrum of the $N = 8$ supersingleton theory. In particular, it was shown that the spectrum, in addition to the $N = 8$ supersingleton states, contains massless states of all higher integer and half-integer spin, and that they fill the irreducible representations of OSp(8,4) [12] with highest spins $s_{max} = 2, 4, 6, \dots$.

In this paper, in addition to providing the necessary background material needed for the derivation of the spectrum [11], we shall present new results. In sects. 2 and

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3, we shall derive the (normal-ordered) generators of the OSp(8, 4) symmetry of the theory, and show that they satisfy the quantum OSp(8, 4) algebra (hence there are no anomalies). Further, in sect. 3, we shall present an unexpected connection between the spectrum of massless states and the work of Vasiliev [13] on higher spin superalgebras, which builds on earlier work by Fradkin and Vasiliev [14]. More precisely, the massless states of our model are in one to one correspondence with the generators of the infinite-dimensional extended higher spin superalgebra, shs(8, 4), of Vasiliev. This suggests that we may have stumbled onto a field-theoretic realization of shs(8, 4). In sect. 4, we shall discuss the issue of the supersingleton–supermembrane connection and provide useful tools for its study. In sect. 5, we comment further on the possible implications of our results to the physics of the $d = 11$ supermembrane, and enumerate open problems. In particular, we suggest that the infinitely many massless states of all higher spin need not contradict cosmological observations if an "inflation" scenario is taken into account. Some useful formulae, and our conventions are collected in the appendices.

2. The action and its symmetries

The $N = 8$ singleton supermultiplet consists of 8 real scalars $\varphi^I (I = 1, \ldots, 8)$, in the 8_v of SO(8), and 8 four-component spinors $\lambda^\alpha_- (\alpha = 1, \ldots, 8)$, in 8_s of SO(8). These fields live on the boundary of AdS_4 which is $S^2 \times S^1$, and therefore depend on coordinates (t, θ, ϕ). In addition to the four-dimensional Majorana condition $\bar{\lambda}_- = \lambda^T_- C$, the spinor λ_- satisfies the following chirality condition

$$\gamma^3 \lambda_- = -\lambda_- , \tag{2.1}$$

which, unlike the usual chirality condition, is compatible with the Majorana condition.

The $N = 8$ supersingleton action is given by [6, 7]

$$\mathscr{L} = -\tfrac{1}{2}\sqrt{-h}\left(h^{ij}\,\partial_i\varphi^I\,\partial_j\varphi^I + \tfrac{1}{4}\varphi^I\varphi^I - i\bar{\lambda}^\alpha_-\gamma^i\,\nabla_i\lambda^\alpha_-\right), \tag{2.2}$$

where $h_{ij} = \mathrm{diag}(-1, 1, \sin^2\theta)$ is the metric, and ∇_i is the covariant derivative on $S^2 \times S^1$. For simplicity in notation, we set the radius of S^2 equal to 1. The action is invariant under the following $N = 8$ supersymmetry transformations

$$\delta\varphi^I = \varepsilon^{\dot{\alpha}}_+ \Sigma^{I\dagger}_{\dot{\alpha}\alpha}\lambda^\alpha_- ,$$

$$\delta\lambda^\alpha_- = -i\gamma^i\partial_i\varphi^I\,\Sigma^I_{\alpha\dot{\alpha}}\varepsilon^{\dot{\alpha}}_+ - \tfrac{1}{2}i\Sigma^I_{\alpha\dot{\alpha}}\varphi^I\varepsilon^{\dot{\alpha}}_- , \tag{2.3}$$

as well as the SO(3,2) × SO(8) transformations

$$\delta\varphi^I = \xi^i_{AB}\,\partial_i\varphi^I + \Omega_{AB}\varphi^I + \Lambda^I{}_J\varphi^J\,,$$

$$\delta\lambda^\alpha_- = \xi^i_{AB}\,\nabla_i\lambda^\alpha_- + \tfrac{1}{4}\gamma^{ij}\left(\nabla_i\xi_{jAB}\right)\lambda^\alpha_- + 2\Omega_{AB}\lambda^\alpha_- + \tfrac{1}{4}\Lambda^{IJ}(\Sigma_{IJ})^\alpha_\beta\lambda^\beta_-\,. \tag{2.4}$$

The parameter $\varepsilon^{\dot\alpha}_\pm = \frac{1}{2}(1 \pm \gamma_3)\varepsilon^{\dot\alpha}(\dot\alpha = 1,\dots,8)$, which is in 8_c of SO(8), is defined by

$$\varepsilon^{\dot\alpha}(t,\theta,\phi) \equiv 2C^{\alpha'\dot\alpha}\varepsilon^{\alpha'}(t,\theta,\phi)\,, \tag{2.5}$$

where $C^{\alpha'\dot\alpha}$ are $4 \times 8 = 32$ arbitrary constant coefficients, and $\varepsilon^{\alpha'}(\alpha' = 1,\dots,4)$ are AdS Killing spinors which therefore satisfy

$$\left(\nabla_a + \tfrac{1}{2}\gamma_a\gamma_3\right)\varepsilon^{\alpha'} = 0\,, \qquad \left(\partial_0 - \tfrac{1}{2}\gamma_0\right)\varepsilon^{\alpha'} = 0\,. \tag{2.6}$$

Here $\nabla_a(a = 1,2)$ is the covariant derivative on S^2. Explicit numerical indices will always denote tangent space indices. Eq. (2.6) implies the 3-covariant equations (suppressing the α' index)

$$\nabla_i\varepsilon_+ - \tfrac{1}{2}\gamma_i\varepsilon_- = 0\,, \qquad \gamma^i\,\nabla_i\varepsilon_- + \tfrac{1}{2}\varepsilon_+ = 0\,. \tag{2.7}$$

One can show that the general solution for eq. (2.6) is given by

$$\varepsilon^{\alpha'}(t,\theta,\phi) = \sqrt{\tfrac{1}{2}}\;e^{1/2\gamma_0 t}\left\{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \begin{pmatrix} -iu_1 \\ iu_2 \end{pmatrix}, \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}, \begin{pmatrix} -iu_3 \\ iu_4 \end{pmatrix}\right\}, \tag{2.8}$$

with

$$u_1 = \begin{pmatrix} iD_{-1/2\,+1/2} \\ D_{+1/2\,+1/2} \end{pmatrix}, \qquad u_2 = \begin{pmatrix} iD_{-1/2\,-1/2} \\ D_{+1/2\,-1/2} \end{pmatrix},$$

$$u_3 = \begin{pmatrix} D_{-1/2\,-1/2} \\ iD_{+1/2\,-1/2} \end{pmatrix}, \qquad u_4 = \begin{pmatrix} -D_{-1/2\,+1/2} \\ -iD_{+1/2\,+1/2} \end{pmatrix}. \tag{2.9}$$

Here we have used the shorthand-notation $D_{mm'} \equiv D^{1/2}_{mm'}(L^{-1})$, where $L(\theta,\phi)$ is the representative of the coset SO(3)/SO(2) = S^2 [15]. In general $D^j_{mm'}(L^{-1})$ denotes the unitary representation matrix of SU(2) for angular momentum j. Some basic properties of these matrices [16] are given in appendix B.

The matrices Σ^I, and $\Sigma^{I\dagger}$ are the SO(8) γ-matrices in a chiral basis, and satisfy

$$\Sigma^I\Sigma^{J\dagger} + I \leftrightarrow J = 2\,\delta^{IJ}\,,$$

$$\Sigma^{I\dagger}\Sigma^J + I \leftrightarrow J = 2\,\delta^{IJ}\,. \tag{2.10}$$

The (conformal) Killing vectors $(\xi^i, \Omega)_{AB} = -(\xi^i, \Omega)_{BA}$ $(A, B = 0,1,2,3,5)$ are the 10 generators of SO(3,2) transformations, and $\Sigma_{IJ} \equiv \Sigma_{[I}\Sigma^{\dagger}_{J]}$. (The explicit form of (ξ^i, Ω) is given in appendix C.) These, and the SO(8) transformation parameter $\Lambda^I{}_J$, satisfy the following equations

$$\nabla_i \xi_{jAB} + \nabla_j \xi_{iAB} = 4\Omega_{AB} h_{ij}, \tag{2.11}$$

$$\nabla^i \partial_i \Omega_{AB} = -\Omega_{AB}, \qquad \partial_i \Lambda^I{}_J = 0. \tag{2.12,2.13}$$

One can easily show that eq. (2.12) is a consequence of eq. (2.11). Furthermore, (from appendix C) we see that $\Omega_{05} = \Omega_{\hat{m}\hat{n}} = 0$ $(\hat{m}, \hat{n} = 1,2,3)$. Therefore ξ_{05}, $\xi_{\hat{m}\hat{n}}$ are the Killing vectors which generate the SO(3) × SO(2) transformations, while $\xi_{0\hat{m}}$, $\xi_{5\hat{m}}$ are conformal Killing vectors which generate the remaining SO(3,2) transformations.

We close this section by giving the commutator algebra of the $N = 8$ supersymmetry transformation rules (2.3). They are

$$[\delta_Q(\varepsilon_1), \delta_Q(\varepsilon_2)] = \delta_{S_O(3,2)}(\xi, \Omega) + \delta_{SO(8)}(\Lambda), \tag{2.14}$$

where

$$\xi^i = -2i\bar{\varepsilon}^{\dot{\alpha}}_{2+}\gamma^i\varepsilon^{\dot{\alpha}}_{1+},$$

$$\Omega = -\tfrac{1}{2}i\left(\bar{\varepsilon}^{\dot{\alpha}}_{2+}\varepsilon^{\dot{\alpha}}_{1-} - 1 \leftrightarrow 2\right),$$

$$\Lambda^{IJ} = -\tfrac{1}{2}i\left(\bar{\varepsilon}_{2+}\Sigma^{[I\dagger}\Sigma^{J]}\varepsilon_{1-} - 1 \leftrightarrow 2\right). \tag{2.15}$$

The dependence on the ten parameters of SO(3,2) can be made explicit by using eq. (2.5). Using the Killing spinor equation (2.11), one can verify that the SO(3,2) and SO(8) parameters given above satisfy the relations (2.11)–(12.13).

3. The quantization of the $N = 8$ supersingleton

In this section we will quantize the $N = 8$ supersymmetric singleton action (2.2). We will first solve the field equations which follow from this action. Then, we will quantize the expansion coefficients occurring in these solutions. Next, we will compute the conserved Noether charges corresponding to the OSp(8,4) transformations. Substituting the solutions into these charges we will obtain an oscillator representation for them. We will then show that these charges satisfy the full *quantum* OSp(8,4) algebra. At the end of this section we shall summarize the spectrum of massless states in the theory and point out an unexpected relationship with the Fradkin–Vasiliev super higher spin algebras.

We begin with the analysis of the field equations which follow from eq. (2.2). They are

$$\partial_i\left(\sqrt{-h}\,h^{ij}\,\partial_j\varphi^I\right)-\tfrac{1}{4}\sqrt{-h}\,\varphi^I=0,\qquad \gamma^i\,\nabla_i\lambda^\alpha_-=0. \qquad (3.1),(3.2)$$

Using the properties of the D-functions (see appendix B) one can verify that a complete solution to these field equations is given by [2,17]

$$\varphi^I=\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left(a^I_{lm}\varphi_{lm}+a^{\dagger I}_{lm}\varphi^*_{lm}\right), \tag{3.3}$$

$$\lambda^\alpha_-=\sum_{j=1/2}^{\infty}\sum_{m=-j}^{+j}\left(d^\alpha_{jm}\lambda_{jm}+d^{\dagger\alpha}_{jm}\lambda^c_{jm}\right), \tag{3.4}$$

with

$$\varphi_{lm}=\frac{1}{\sqrt{4\pi}}\,\mathrm{e}^{i(l+1/2)t}D^l_{0m}(L^{-1}), \tag{3.5}$$

and

$$\lambda_{jm}=\begin{pmatrix}u_{jm}\\ iu_{jm}\end{pmatrix},\quad u_{jm}=\mathrm{e}^{-\pi i/4}\left(\frac{2j+1}{8\pi}\right)^{1/2}\mathrm{e}^{-i(j+1/2)t}\begin{pmatrix}D^j_{-1/2\,m}(L^{-1})\\ D^j_{+1/2\,m}(L^{-1})\end{pmatrix}. \tag{3.6}$$

Note that the single valuedness of the scalar field requires that we work on double (more generally even) covering of AdS_4. The solutions are normalized such that

$$\langle\varphi_{lm},\varphi_{l'm'}\rangle\equiv i\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sin\theta\,\varphi^*_{lm}\,\overleftrightarrow{\partial}_0\varphi_{l'm'}=\delta_{ll'}\,\delta_{mm'}, \tag{3.7}$$

$$\langle\lambda_{jm},\lambda_{j'm'}\rangle\equiv\int \mathrm{d}\,\theta\,\mathrm{d}\,\phi\,\sin\theta\,\bar\lambda_{jm}\gamma^0\lambda_{j'm'}=\delta_{jj'}\,\delta_{mm'}. \tag{3.8}$$

We now proceed with the canonical quantization of the model. We impose the following (anti-) commutation relations

$$[a^I_{lm},a^{\dagger J}_{l'm'}]=\delta^{IJ}\delta_{ll'}\delta_{mm'},\qquad l=0,1,\dots,\qquad -l\leqslant m\leqslant l, \tag{3.9}$$

$$\{d^\alpha_{jm},d^{\dagger\beta}_{j'm'}\}=\delta^{\alpha\beta}\delta_{jj'}\delta_{mm'},\qquad j=\tfrac{1}{2},\tfrac{3}{2},\dots,\qquad -j\leqslant m\leqslant j. \tag{3.10}$$

Other (anti)commutators vanish. The a_{lm} and d_{jm} are now operators in a Fock space whose vacuum $|0\rangle$ is defined by

$$a_{lm}|0\rangle=d_{jm}|0\rangle=0. \tag{3.11}$$

We now turn to the calculation of the conserved OSp(8, 4) Noether charges. They are obtained from the conserved Noether currents defined by

$$J^i = \frac{\partial \mathscr{L}}{\partial \partial_i \varphi^I} \delta\varphi^I + \frac{\partial_{\mathrm{R}} \mathscr{L}}{\partial \partial_i \lambda^\alpha} \delta\lambda^\alpha - K^i , \tag{3.12}$$

where ∂_{R} denotes differentiation from the right, and K^i is defined by

$$\delta\mathscr{L} = \partial_i K^i . \tag{3.13}$$

It is clear that $\partial_i J^i = 0$ by virtue of the (Euler–Lagrange) equations of motion.

Applying the formula (3.12) to the lagrangian (2.2), we obtain the following result

$$J^i_{AB} = \sqrt{-h}\, h^{ij}\Big(T_{jk}\xi^k_{AB} + \tfrac{1}{2}\varphi^2 \overleftrightarrow{\partial}_j \Omega_{AB}\Big) , \tag{3.14}$$

$$J^i_{IJ} = \sqrt{-h}\, h^{ij}\varphi_{[I}\, \partial_j \varphi_{J]} + \tfrac{1}{4} i\sqrt{-h}\, \bar{\lambda}_- \gamma^i \Sigma_{IJ} \lambda_- , \tag{3.15}$$

$$J^i_{\alpha'\dot\alpha} = i\sqrt{-h}\, \bar{\lambda}^\alpha_- \gamma^i \Sigma^I_{\alpha\dot\alpha}\Big[\partial_j \varphi_I \gamma^j \varepsilon_{\alpha'+} + \tfrac{1}{2}\varphi_I \varepsilon_{\alpha'-}\Big] , \tag{3.16}$$

where the energy–momentum tensor is given by

$$T_{ij} = -\partial_i \varphi^I \partial_j \varphi_I + \tfrac{1}{2} h_{ij}\big(\partial_k \varphi^I \partial^k \varphi_I + \tfrac{1}{4}\varphi^2\big) - \tfrac{1}{2} i\bar{\gamma}_- \gamma_{(i} \nabla_{j)} \lambda_- . \tag{3.17}$$

The conserved charges corresponding to these currents are given by

$$\begin{aligned} M_{AB} &= \int \mathrm{d}\theta\, \mathrm{d}\phi\, J^{i=0}_{AB} , \\ T_{IJ} &= \int \mathrm{d}\theta\, \mathrm{d}\phi\, J^{i=0}_{IJ} , \\ \vartheta_{\alpha'\dot\alpha} &= \int \mathrm{d}\theta\, \mathrm{d}\phi\, J^{i=0}_{\alpha'\dot\alpha} . \end{aligned} \tag{3.18}$$

We now substitute the solutions (3.3) and (3.4) for φ^I and λ^α_- into the expressions (3.18) for the Noether charges. Using several properties of the D-functions we thus obtain after a tedious but straightforward calculation the following oscillator representations. The SO(3,2) charges are given by

$$M_{05} = \sum_{l,m} \left(l + \tfrac{1}{2}\right) a^{\dagger I}_{lm} a^{I}_{lm} + \sum_{j,m} \left(j + \tfrac{1}{2}\right) d^{\dagger\alpha}_{jm} d^{\alpha}_{jm} + c\,, \tag{3.19}$$

$$M_{12} = \sum_{l,m} m a^{\dagger I}_{lm} a^{I}_{lm} + \sum_{j,m} m\, d^{\dagger\alpha}_{jm} d^{\alpha}_{jm}\,, \tag{3.20}$$

$$\begin{aligned} M_{23} + iM_{31} = &\sum_{l,m} \left[(l-m)(l+m+1)\right]^{1/2} a^{\dagger I}_{l,m+1} a^{I}_{lm} \\ &+ \sum_{j,m} \left[(j-m)(j+m+1)\right]^{1/2} d^{\dagger\alpha}_{j,m+1} d^{\alpha}_{jm}\,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} iM_{03} + M_{53} = &\sum_{l,m} \left[(l-m+1)(l+m+1)\right]^{1/2} a^{\dagger I}_{l+1,m} a^{I}_{lm} \\ &+ \sum_{j,m} \left[(j-m+1)(j+m+1)\right]^{1/2} d^{\dagger\alpha}_{j+1,m} d^{\alpha}_{jm}\,, \end{aligned} \tag{3.22}$$

$$\begin{aligned} (iM_{01} + M_{51}) + i(iM_{02} + M_{52}) = &- \sum_{l,m} \left[(l+m+2)(l+m+1)\right]^{1/2} a^{\dagger I}_{l+1,m+1} a^{I}_{lm} \\ &- \sum_{j,m} \left[(j+m+2)(j+m+1)\right]^{1/2} d^{\dagger\alpha}_{j+1,m+1} d^{\alpha}_{jm}\,, \end{aligned} \tag{3.23}$$

$$\begin{aligned} (iM_{01} + M_{51}) - i(iM_{02} + M_{52}) = &\sum_{l,m} \left[(l-m+2)(l-m+1)\right]^{1/2} a^{\dagger I}_{l+1,m-1} a^{I}_{lm} \\ &+ \sum_{j,m} \left[(j-m+2)(j-m+1)\right]^{1/2} d^{\dagger\alpha}_{j+1,m-1} d^{\alpha}_{jm}\,. \end{aligned} \tag{3.24}$$

The constant c in eq. (3.19) is a consequence of normal-ordering ambiguity in the definition of M_{05}. It is the sum of zero-point energies of the oscillators, and is given by

$$c = \tfrac{1}{2} \sum_{l,m} \left(l + \tfrac{1}{2}\right) - \tfrac{1}{2} \sum_{j,m} \left(j + \tfrac{1}{2}\right). \tag{3.25}$$

These sums are divergent, and need regularization. However, it is more satisfactory to determine c by demanding closure of the quantum algebra, which we will do below. Note that there is no normal ordering ambiguity in the definition of M_{12}. This is because $\sum_{m=-l}^{+l} m = 0$ identically. The other operators are well defined since they consist of oscillators which commute with each other. For the supersymmetry charges we find

$$Q^{\dot{\alpha}1} = \sum_{l,m} \left[(l+m+1)^{1/2} a_{lm}^{\dagger I} d_{l+1/2,\, m+1/2}^{\beta} + (l+m)^{1/2} d_{l-1/2,\, m-1/2}^{\dagger\beta} a_{lm}^{I} \right] (\Sigma_I)_{\beta}{}^{\dot{\alpha}}, \tag{3.26}$$

$$Q^{\dot{\alpha}2} = \sum_{l,m} \left[(l-m+1)^{1/2} a_{lm}^{\dagger I} d_{l+1/2,\, m-1/2}^{\beta} + (l-m)^{1/2} d_{l-1/2,\, m+1/2}^{\dagger\beta} a_{lm}^{I} \right] (\Sigma_I)_{\beta}{}^{\dot{\alpha}}, \tag{3.27}$$

where $Q^{\dot{\alpha}1}$ and $Q^{\dot{\alpha}2}$ are given by the following combination of supercharges

$$Q^{\dot{\alpha}1} = \tfrac{1}{4}\left(i\vartheta^{\dot{\alpha}1} + \vartheta^{\dot{\alpha}2} - \vartheta^{\dot{\alpha}3} - i\vartheta^{\dot{\alpha}4} \right), \tag{3.28}$$

$$Q^{\dot{\alpha}2} = \tfrac{1}{4}\left(\vartheta^{\dot{\alpha}1} + i\vartheta^{\dot{\alpha}2} + i\vartheta^{\dot{\alpha}3} + \vartheta^{\dot{\alpha}4} \right). \tag{3.29}$$

Finally, we find for the SO(8) charges the following oscillator representation

$$T^{IJ} = 2i \sum_{lm} a_{lm}^{\dagger[I} a_{lm}^{J]} + \tfrac{1}{2} i \sum_{jm} d_{jm}^{\dagger\alpha} \Sigma_{\alpha\beta}^{IJ} d_{jm}^{\beta}. \tag{3.30}$$

We are now able to calculate the quantum algebra. We find that for $c=0$ the charges defined above satisfy the following algebra

$$[M_{AB}, M_{CD}] = -i(\eta_{BC} M_{AD} - \eta_{AC} M_{BD} - \eta_{BD} M_{AC} + \eta_{AD} M_{BC}), \tag{3.31}$$

$$[T_{IJ}, T_{KL}] = i(\delta_{JK} T_{IL} - \delta_{IK} T_{JL} - \delta_{JL} T_{IK} + \delta_{IL} T_{JK}), \tag{3.32}$$

$$\{ Q^{\dot{\alpha}}, \overline{Q}^{\dot{\beta}} \} = \delta^{\dot{\alpha}\dot{\beta}} l^{AB} M_{AB} + \tfrac{1}{2} (\Sigma_{IJ})^{\dot{\alpha}\dot{\beta}} T^{IJ}, \tag{3.33}$$

$$[M_{AB}, Q^{\dot{\alpha}}] = \tfrac{1}{2} i l_{AB} Q^{\dot{\alpha}}, \qquad [T^{IJ}, Q^{\dot{\alpha}}] = -\tfrac{1}{2} i (\Sigma^{IJ})_{\dot{\beta}}^{\dot{\alpha}} Q^{\dot{\beta}}. \tag{3.34),(3.35}$$

Here $l^{r5} = \tilde{\gamma}^r$, $l^{rs} = \tilde{\gamma}^{rs} (r, s = 0,1,2,3)$, where we have used the following γ-matrix notation

$$\tilde{\gamma}_0 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \tilde{\gamma}_i = \begin{pmatrix} 0 & -i\sigma^i \\ i\sigma^i & 0 \end{pmatrix}, \qquad \tilde{C} = \tilde{\gamma}_0 \tilde{\gamma}_2. \tag{3.36}$$

This representation is chosen so that the Majorana spinor $Q^{\dot\alpha}$ has the simple form

$$Q^{\dot\alpha} = \begin{pmatrix} Q^{\dot\alpha 1} \\ Q^{\dot\alpha 2} \\ (Q^{\dot\alpha 2})^{\dagger} \\ (-Q^{\dot\alpha 1})^{\dagger} \end{pmatrix}. \tag{3.37}$$

The above results show that there are no local anomalies in any of the symmetries of the $N=8$ supersingleton model.

In the case of string theories the zeta-function regularization gives the same value of c as that determined by the closure of the Lorentz algebra. It is interesting to see whether this is also the case in our model. The zeta-function regularization gives

$$\begin{aligned} c_{reg} &\equiv \lim_{s\to -1}\left\{ \tfrac{1}{2}\sum_{l,m}\left(l+\tfrac{1}{2}\right)^{-s} - \tfrac{1}{2}\sum_{j,m}\left(j+\tfrac{1}{2}\right)^{-s}\right\}, \\ &= \left(-\tfrac{3}{4}-1\right)\zeta(-2), \end{aligned} \tag{3.38}$$

where $-\frac{3}{4}$ and -1 are bosonic and fermionic contributions respectively. Since $\zeta(-2)=0$ we obtain $c_{\rm reg}=0$ [18], which coincides with the value determined by the closure of the OSp(8,4) algebra. It is remarkable that the zero-point energies sum up to zero for bosons and fermions *separately*.

Using the results of sect. 2 and 3, elsewhere [11] we have studied the spectrum of the $d=4$, $N=8$ supersingleton theory. The singleton states are rather unconventional. As Dirac noted first [1], their wave function has a fixed dependence on the radial coordinate of AdS_4. Thus, there is no principal quantum number associated with this radial direction. The two singleton states (obtained by the action of two singleton creation operators on the vacuum) yield infinitely many massless states whose quantum numbers are exhibited in table 1. In this table s denotes the spin of the massless state, and L is the label of the supermultiplet in which a given massless state occurs. The table clearly shows that there is a regular pattern which repeats itself modulo units of spin 2 (except for spin 0 states which are special); i.e. the SO(8) content of spin s states is the same as that of spin $(s+2)$ states.

The occurrence of infinitely many massless higher spin particles in our model implies the existence of infinitely many (local) gauge symmetries which are analogous to the Maxwell, general-coordinate and local supersymmetries, associated with spins 1, 2 and $\frac{3}{2}$, respectively. In that case one would expect that the massless states tabulated above gauge an infinite-dimensional superalgebra. In fact, we have found a remarkable relationship between these massless states and the extended higher spin superalgebra shs(8,4) of Vasiliev [13] which supports this expectation. More

TABLE 1

$L \backslash s$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	3	$\frac{7}{2}$	...	$2s$	$2s+\frac{1}{2}$	$2s+1$	$2s+\frac{3}{2}$	...
-1	70	56	28	8	1									
0	2	8	28	56	70	56	28	8	...					
1					1	8	28	56	...					
⋮														
$s-2$									...	1				
$s-1$									...	70	56	28	8	...
s										1	8	28	56	...
⋮														

specifically, Vasiliev's shs(8,4) algebra has the generators which carry spin s, and are the kth rank antisymmetric tensors of SO(8) as follows

$$s = 2, 4, 6, \dots, \qquad k = 0, 4, 8,$$

$$s = 1, 3, 5, \dots, \qquad k = 2, 6,$$

$$s = \tfrac{3}{2}, \tfrac{7}{2}, \dots, \qquad k = 1, 5,$$

$$s = \tfrac{5}{2}, \tfrac{9}{2}, \dots, \qquad k = 3, 7.$$

Remarkably, both the spin and the SO(8) content of these generators agree precisely with those of the massless states given in table 1. (The comparison is to be made for gauge fields, of course, i.e. fields with $s \geqslant 1$.) This suggests that the $N = 8$ supersingleton model provides a field-theoretic realization of shs(8,4), which in turn has an application in a consistent description of higher spin gauge-fields interactions. In fact, Fradkin and Vasiliev [19] have already shown that consistent cubic interactions of higher spin fields which gauge shs(8,4) do exist in AdS_4. It would be interesting to see whether shs(8,4) can be realized as the algebra of the stress tensor of the $d = 4$, $N = 8$ supersingleton theory. Some progress is made in this direction in ref. [20].

4. Singletons and membranes

So far we have discussed the $N = 8$ supersingleton theory in its own right. We devote this section to a discussion of possible connections between the $N = 8$ supersingleton theory [6,7] and the $d = 11$ supermembrane [8]. It has been conjectured [9–11,21] that the $N = 8$ supersingleton theory may arise from the $d = 11$ supermembrane action in a physical gauge. One consequence of this would be that the two singleton states could be interpreted as the massless excitations of a membrane [11].

The $d = 11$ supermembrane in $AdS_4 \times S^7$ background, where the 4-form field strength of $d = 11$ supergravity assumes the value of the AdS_4 volume form, was considered in ref. [9]. In an attempt to solve the membrane field equations the membrane world-volume was identified with the $S^2 \times S^1$ boundary of AdS_4. Below, we will see, however, that in a background-field expansion around this candidate solution, the term linear in the fluctuations (i.e. the tadpole term) actually diverges on the boundary of AdS_4. A modified version of this candidate solution which avoids this problem has been recently found, although it does not seem to give rise to the full $N = 8$ supersingleton theory [10]. Therefore, a rigorous connection between the $N = 8$ supersingletons and the $d = 11$ supermembrane has not been established yet*. However, recent work [6, 7] suggests that such connection may exist. This is mainly because there is a natural way in which the supersingletons (described by a field theory of scalars and spinors on the boundary of an AdS space) can be associated [6, 7] with super p-branes [8, 22, 23]. Firstly, the super AdS groups exist only in dimensions $d \leqslant 7$ [24]. In this case the boundary of the AdS space is $S^p \times S^1$, $p = 1, \dots, 5$. These are precisely the values of p for which super p-branes exist [23]. Secondly, the internal symmetry groups occurring in the super AdS groups are the isometry groups of S^1, S^3 and S^7. Adding up the dimension of AdS space with the dimension of the appropriate internal space, remarkably, one obtains the critical dimensions for super p-branes [7].

Given the motivation above, in the remainder of this section we shall firstly discuss the supersymmetry of a membrane propagating in a given $d = 11$ supergravity background, and secondly provide an action formula for small fluctuations around an arbitrary background.

The $d = 11$ supermembrane action is given by [8]

$$
\begin{aligned}
&S = -\frac{1}{\alpha'^3} \int \mathrm{d}\tau \,\mathrm{d}\sigma \,\mathrm{d}\rho \left(\sqrt{-h} + \tfrac{1}{3}\varepsilon^{ijk}\, \partial_i Z^\Lambda\, \partial_j Z^\Sigma\, \partial_k Z^\Omega B_{\Omega\Sigma\Lambda}\right), \\
&h_{ij} = \Pi_i^a \Pi_j^b \eta_{ab}, \qquad i, j = 0,1,2 \quad a, b = 0,1,\dots,10, \\
&\Pi_i^a = \partial_i Z^\Lambda E_\Lambda^a, \qquad Z^\Lambda = (X^M, \Theta^{\hat\alpha}), \qquad M = 0,1,\dots,10, \qquad \hat\alpha = 1,2,\dots,32,
\end{aligned}
\tag{4.1}
$$

where (τ, σ, ρ) are the coordinates on the world-volume with metric h_{ij}, $E_\Lambda^A(X^M, \Theta)$

* In a previous letter [11], the relation between the $d = 4$, $N = 8$ supersingleton theory and the $d = 11$ supermembrane theory was incorrectly asserted to have been established. (Since ref. [11] dealt with the $N = 8$ supersingleton theory in its own right, none of the results in that letter were affected by this assertion.)

is the supervielbein and $B_{\Omega\Sigma\Lambda}(X^M, \Theta)$ is the super three-form potential appropriate to the eleven-dimensional superspace. $X^M(\tau, \sigma, \rho)$ are the bosonic coordinates and $\Theta(\tau, \sigma, \rho)$ are the fermionic coordinates of eleven-dimensional superspace. Thus Θ is a 32-component Majorana spinor. The parameter α' is related to the membrane tension T as $T = 1/\alpha'^3$*. T has dimension three, i.e. $T \sim M^3$ where M is some mass unit. Note that the world-volume coordinates are dimensionless, while $X^M \sim M^{-1}$, and $\Theta \sim M^{-1/2}$. (For further conventions see appendix A. For superspace conventions see ref. [25].)

The supermembrane action (4.1) is in variant under κ-symmetry provided that the background fields (i.e. the supervielbein and the super 3-form) satisfy certain constraints which are equivalent to the equations of motion of eleven-dimensional supergravity. Thus, in order to find a classical solution to the eleven-dimensional supermembrane theory, one must first find a background which solves the $d = 11$ supergravity equations of motion. Next, one must solve the supermembrane equations of motion which follow from eq. (4.1) in that background.

In a background in which all the fermionic fields are zero, the local κ and local ϵ supersymmetry transformation rules of the fermionic fields read

$$\delta\Psi_M = \tilde{D}_M\epsilon \equiv \left(\nabla_M + \tfrac{1}{288}\left(\Gamma^{PQRS}\Gamma_M - 3\Gamma_M\Gamma^{PQRS}\right)H_{PQRS}(x)\right)\epsilon\,, \tag{4.2}$$

$$\delta\Theta = (1 + \bar{\Gamma})\kappa + \epsilon\,, \tag{4.3}$$

where Ψ is the $d = 11$ gravitino, $H_{PQRS}(x) = 4\,\partial_{[P}B_{QRS]}(x)$, x^M is the background value of X, and $\bar{\Gamma}$ is the background value of

$$\Gamma = \frac{1}{6\sqrt{-h}}\varepsilon^{ijk}\Pi_i^a\Pi_j^b\Pi_k^c\Gamma_{abc}\,. \tag{4.4}$$

Thus, the criteria for supersymmetry of a solution in which all the fermionic fields vanish is [9]

$$\delta\Theta = 0 \Rightarrow (1 - \bar{\Gamma})\epsilon = 0\,, \tag{4.5}$$

$$\delta\Psi_M = 0 \Rightarrow \tilde{D}_M\epsilon = 0\,. \tag{4.6}$$

In order to make contact with the Killing spinors on the $S^2 \times S^1$ boundary of AdS_4 satisfying eq. (2.6), we must choose a background such that eq. (4.6) implies the AdS_4 Killing spinor equation. One way of achieving this is to choose the $d = 11$

* In $d = 11$, the Hilbert–Einstein term in the effective lagrangian must be proportional to α'^{-9} on dimensional grounds. Thus, in an $AdS_4 \times M^7$ background where M^7 is a seven dimensional Einstein manifold of characteristic size a^{-1}, the product of α'^{-9} with the volume of M^7, which is proportional to a^{-7}, should be identified with the inverse square of the usual four-dimensional gravitational coupling constant κ. Hence we have the relation $\kappa^2 \sim \alpha'^9 a^7$.

background spacetime to be a product of an AdS_4 of inverse radius a with a 7-dimensional Einstein manifold of inverse radius $\frac{1}{2}a$, and choose $H_{\mu\nu\rho\sigma} = \frac{3}{2}a\sqrt{-g}\,\varepsilon_{\mu\nu\rho\sigma}$, where g is the determinant of the AdS_4 metric $g_{\mu\nu}(x)$ [26, 27]. We use a coordinate system in which the AdS_4 metric is given by

$$\mathrm{d}s^2 = (a\cos\beta)^{-2}\left[-\mathrm{d}t^2 + \mathrm{d}\beta^2 + \sin^2\beta\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\right)\right]. \tag{4.7}$$

The angular variables θ and ϕ satisfy the usual constraints, while $0 \leqslant \beta \leqslant \frac{1}{2}\pi$, and $\beta = \frac{1}{2}\pi$ corresponds to spatial infinity. In this background, in order to solve eq. (4.6) it is appropriate to make the following ansatz

$$\epsilon^{\alpha'\dot{\alpha}} = \epsilon^{\alpha'}(x) \otimes \eta^{\dot{\alpha}}(y), \qquad \alpha' = 1, 2, \ldots, 4, \qquad \dot{\alpha} = 1, 2, \ldots, 8, \tag{4.8}$$

where $\epsilon^{\alpha'\dot{\alpha}}$ are spinors of eleven dimensions. The index α' labels a spinor of SO(3, 2), and $\dot{\alpha}$ labels the appropriate representation(s) of the isometry group of the internal manifold (e.g. the 8_c of SO(8) for 7-sphere). (We omit the SO(10, 1) spinor index of $\epsilon^{\alpha'\dot{\alpha}}$, the SO(3, 1) spinor index of $\epsilon^{\alpha'}$ and the SO(7) spinor index of $\eta^{\dot{\alpha}}$.) From this ansatz, and eq. (4.6) it follows that $\epsilon^{\alpha'}$ are Killing spinors on AdS_4 satisfying [27]

$$\left(\nabla_\mu - \tfrac{1}{2}a\gamma_\mu\right)\epsilon^{\alpha'} = 0, \tag{4.9}$$

where ∇_μ is the usual Lorentz covariant derivative on AdS_4. One can show that the general solution for $\epsilon^{\alpha'}$ is given by [9, 28]

$$\epsilon^{\alpha'} = \sqrt{2}\,(a\cos\beta)^{-1/2}\left(\cos\tfrac{1}{2}\beta + \gamma_3\sin\tfrac{1}{2}\beta\right)\varepsilon^{\alpha'}(t, \theta, \phi), \tag{4.10}$$

where $\varepsilon^{\alpha'}(t, \theta, \phi)$ are precisely the Killing spinors on the $S^2 \times S^1$ boundary of AdS_4 which satisfy eq. (2.6), and whose solutions are given in eq. (2.8). One must still solve eq. (4.5), which certainly depends on the specific details of the solution to the membrane equations of motion, as well as eq. (4.6) in the internal directions. If one chooses the internal manifold to be a 7-sphere, then eq. (4.6) implies [27]

$$\left(\nabla_m + \tfrac{1}{4}ia\gamma_m\right)\eta^{\dot{\alpha}} = 0, \tag{4.11}$$

where ∇_m is the standard Lorentz-covariant derivative on the 7-sphere.

We now examine the small fluctuations around an arbitrary classical solution. To this end, we define a normal coordinate expansion for the coordinates X^M in the standard way [29]

$$X^M = x^M + (\alpha')^{3/2}\xi^M + \mathrm{O}(\xi^2), \tag{4.12}$$

where x^M is the background value for X^M and ξ^M is the normal coordinate. Using eq. (4.12) one obtains the following useful expansion formulae (in this formula we suppress α')

$$\partial_i X^M = \partial_i x^M + \nabla_i \xi^M - \tfrac{1}{3}\partial_i x^N R^M_{PNQ}\xi^P\xi^Q + \mathrm{O}(\xi^3),$$

$$g_{MN}(X) = g_{MN}(x) - \tfrac{1}{3}R_{MPNQ}\xi^P\xi^Q + \mathrm{O}(\xi^3),$$

$$\begin{aligned} B_{MNP} &= B_{MNP}(x) + \xi^Q \nabla_Q B_{MNP}(x) \\ &\quad + \tfrac{1}{2}\xi^Q\xi^R\big(\nabla_Q\nabla_R B_{MNP}(x) + R^S_{QR[M}B_{NP]S}(x)\big) + \mathrm{O}(\xi^3), \end{aligned} \tag{4.13}$$

where $\nabla_i\xi^M$ is the usual covariant derivative. For the induced metric which is defined by $h_{ij} = \partial_i X^M \partial_j X^N g_{MN}(X)$ we obtain the following expansion

$$\begin{aligned} h_{ij} &= \bar{h}_{ij} + 2\,\partial_{(i}x^M \nabla_{j)}\xi^N g_{MN}(x) \\ &\quad + \nabla_i\xi^M \nabla_j\xi^N g_{MN}(x) - \partial_i x^M \partial_j x^N R_{MPNQ}\xi^P\xi^Q + \mathrm{O}(\xi^3), \end{aligned} \tag{4.14}$$

with $\bar{h}_{ij} = \partial_i X^M \partial_j x^N g_{MN}(x)$. Applying these formulae to the expansion of the supermembrane action (4.1) we obtain

$$\mathscr{L} = \sum_{n=0}^{\infty} a'^{3(n-2)/2}\mathscr{L}^n\ , \tag{4.15}$$

where

$$\mathscr{L}^0 = -\sqrt{-\bar{h}} + \tfrac{1}{3}\epsilon^{ijk}\,\partial_i x^M \partial_j x^N \partial_k x^P B_{MNP}(x),$$

$$\mathscr{L}^1 = -\sqrt{-\bar{h}}\,\bar{h}^{ij}\,\partial_i x^M \nabla_j\xi^N g_{MN}(x) + \tfrac{1}{3}\varepsilon^{ijk}\xi^M \partial_i x^N \partial_j x^P \partial_k x^Q H_{MNPQ}(x),$$

$$\begin{aligned} \mathscr{L}^2 &= -\tfrac{1}{2}\sqrt{-\bar{h}}\ \nabla_i\xi^M \nabla^i\xi^N g_{MN} + \tfrac{1}{2}\sqrt{-\bar{h}}\ \partial_i x^M \partial_j x^N \bar{h}^{ij} R_{MPNQ}\xi^P\xi^Q \\ &\quad - \tfrac{1}{2}\sqrt{-\bar{h}}\ \nabla^i\xi_M \nabla^j\xi_N\big(\partial_i x^M \partial_j x^N - \partial_j x^M \partial_i x^N - \bar{h}_{ij}\,\partial_k x^M \partial^k x^N\big) \\ &\quad + \tfrac{1}{2}\varepsilon^{ijk}\xi^M \nabla_i\xi^N \partial_j x^P \partial_k x^Q H_{MNPQ} - \tfrac{1}{6}\varepsilon^{ijk}\xi^M\xi^N \partial_i x^P \partial_j x^Q \partial_k x^R \nabla_N H_{PQRM} \\ &\quad + i\sqrt{-\bar{h}}\,\bar{h}^{ij}\,\partial_i x^M E^{\hat{a}}_M \bar{\Theta}\Gamma_{\hat{a}}(1-\bar{\Gamma})\,\tilde{\nabla}_j\Theta. \end{aligned} \tag{4.16}$$

In deriving the Θ-dependent terms in eq. (4.16) we have assumed that the background values of all fermions are zero. (In view of this we denote the fluctuation of Θ by the same symbol.) Furthermore, we have used

$$\Pi_i^{\hat{\alpha}} = \left(\tilde{\nabla}_i \Theta\right)^{\hat{\alpha}} + \mathrm{O}(\Theta^3), \tag{4.17}$$

where

$$\tilde{\nabla}_i = \partial_i + \partial_i x^M \left[\tfrac{1}{4}\omega_M^{ab}(x)\Gamma_{ab} + \tfrac{1}{288}\left(\Gamma^{PQRS}\Gamma_M - 3\Gamma_M\Gamma^{PQRS}\right)H_{PQRS}(x)\right]. \tag{4.18}$$

The ansatz for x^M suggested in ref. [9] in order to solve the brane-wave equation which follows from eq. (4.1) was: $t=\tau$, $\theta=\sigma$, $\phi=\rho$, $\beta=\frac{1}{2}\pi$, $\partial_i y^m = 0$ ($m=1,\dots,7$). However, an appropriate rescaling of the fluctuations to render $\mathscr{L}^2$ independent of β, also has the effect of making the tadpole term $\mathscr{L}^1$ diverge as $(\cos\beta)^{-1/2}$. This is unacceptable. Recently, however, an alternative ansatz has been found [10] in which $\mathscr{L}^1$ vanishes identically for all values of β. The details of this solution and the issue of singletons will be discussed in ref. [10].

5. Comments

If the conjectured supersingleton–supermembrane connection actually exists, then we should interpret the states obtained by repeated action of an arbitrary number of singleton creation operators on the vacuum to be corresponding to excitations of a supermembrane. In the $N=8$ supersingleton theory, although interactions on the world-volume are not possible [7], we may still speak about interactions of spacetime (AdS_4) fields. (One constructs the membrane propagator and vertex operators. Using these, one can build spacetime amplitudes in an operator formalism in much the same way as is done for string theories in a light-cone operator formalism.) It is not known at present what kind of interactions (of spacetime fields), if any, would be obtained in this way*. Moreover, it is not clear how to incorporate the membrane interactions via a change in the topology of the world-volume, since in our model the world-volume has a fixed topology (i.e. $S^2 \times S^1$). In any event, it would be interesting to see if the consistent cubic interactions of Fradkin and Vasiliev [19] could be obtained from membrane amplitudes.

* An alternative view has been advocated by Flato and Fronsdal [30]. They consider the massless states to be the singleton bound states, and argue that in order to obtain meaningful interactions of these states in the whole of AdS space, one should quantize the singletons with an unusual spin-statistics.

Note that the spectrum of massless states, in addition to the usual SO(8) gauge fields, gravitini and graviton, contains 28 new gauge fields, 56 new gravitini and $35+35+1$ new graviton fields, as well as higher spin massless states. Before deciding whether the new states are physically acceptable, one should first analyse their couplings (if any) to the usual massless particles of spin $\leqslant 2$. Using the result of this paper, assuming that the conjectured supersingleton–supermembrane connection exists, one could contemplate computing the vertex operators in our model, in an attempt to understand the nature of the higher spin massless fields. One might expect that the couplings among particles which will survive in a limit in which the membrane tension goes to zero while keeping the gravitational coupling fixed, would be those described in the de Wit–Nicolai's $N=8$ supergravity [31].

Concerning the cosmological implications of the infinitely many massless higher spin particles, one can envisage an inflationary scenario in which all these states are diluted sufficiently enough not to violate any cosmological observations. For a discussion of how this works in the case of the graviton, see, for example ref. [32], and for other massless particles with gravitational coupling, see ref. [33]. Although inflation scenarios usually favor de Sitter space, there does exist a mechanism for triggering inflation in anti-de Sitter space. In fact, one such mechanism has been suggested [34] for de Wit–Nicolai's $N=8$ supergravity.

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Appendix A

CONVENTIONS

$$\operatorname{sign} h_{ij} = (-,+,+), \qquad \operatorname{sign}\eta_{ab} = (-,+,+,+,+,+,+,+,+,+,+),$$

$$\{\Gamma^a, \Gamma^b\} = 2\eta_{ab}, \qquad a = 0,1,\ldots,10,$$

$$\Gamma^{a\mathrm{T}} = -C^{-1}\Gamma^a C, \qquad \bar{\psi} = \psi^\dagger \Gamma_0, \qquad \psi^c = C\bar{\psi}^{\mathrm{T}}, \qquad C^{\mathrm{T}} = -C,$$

$$\Gamma^\mu = i\gamma_5\gamma^\mu \otimes 1, \qquad \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0, \qquad \mu = 0,1,2,3,$$

$$\Gamma^{m+3} = \gamma_5 \otimes \gamma^m, \qquad \gamma^{m\dagger} = \gamma^m, \qquad m = 1,2,\ldots,7,$$

$$\gamma_5 = i\gamma_{0123}, \qquad \gamma_5^\dagger = \gamma_5, \qquad \gamma_5^2 = 1,$$

$$\gamma_i = -\sigma_1 \otimes \tau_i, \qquad \gamma_3 = -\sigma_2 \otimes 1, \qquad C = \gamma_{13} \otimes 1,$$

$$\tau_0 = i\sigma_3, \qquad \tau_1 = -\sigma_2, \qquad \tau_2 = \sigma_1, \qquad \varepsilon_{012} = \varepsilon_{0123} = 1,$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \qquad \bar{\psi} \equiv \psi^\dagger \gamma_0 = (-i\psi_3^*, i\psi_4^*, -i\psi_1^*, i\psi_2^*).$$

Appendix B

SOME PROPERTIES OF $D^j_{mm'}$

$$\left[D^j_{mm'}(L)\right]^* = D^j_{m'm}(L^{-1}),$$

$$\nabla_{\pm} D^j_{mm'}(L^{-1}) = -i\left[(j \mp m)(j \pm m + 1)\right]^{1/2} D^j_{m\pm 1 m'}(L^{-1}),$$

$$\nabla_{\mp} \equiv \nabla_1 \pm i\,\nabla_2,$$

$$\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sin\theta D^j_{mm'}(L)\,D^{j'}_{m'm''}(L^{-1}) = \frac{4\pi}{2j+1}\,\delta_{jj'}\,\delta_{mm''} \qquad (\text{no sum over } m'),$$

$$\int \mathrm{d}\theta\,\mathrm{d}\phi\,\sin\theta D^{j_1}_{m_1 n_1}(L)\,D^{j_2}_{m_2 n_2}(L)\,D^{j_3}_{m_3 n_3}(L) = 4\pi \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ n_1 & n_2 & n_3 \end{pmatrix},$$

$$(m_1 + m_2 + m_3 = 0),$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix},$$

$$= (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix},$$

$$\begin{pmatrix} j+\frac{1}{2} & j & \frac{1}{2} \\ m & -m-\frac{1}{2} & \frac{1}{2} \end{pmatrix} = (-1)^{j-m-1/2} \left[\frac{j-m+\frac{1}{2}}{(2j+2)(2j+1)}\right]^{1/2}.$$

Appendix C

(CONFORMAL) KILLING VECTORS ON $S^2 \times S^1$

$$\xi_{05} = \frac{\partial}{\partial t}, \qquad \xi_{12} = \frac{\partial}{\partial\phi}, \qquad \xi_{23} + i\xi_{31} = \mathrm{e}^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right),$$

$$\xi_{05} = \frac{\partial}{\partial t}, \qquad \xi_{12} = \frac{\partial}{\partial\phi}, \qquad \xi_{23} + i\xi_{31} = \mathrm{e}^{i\phi}\left(i\frac{\partial}{\partial\theta} - \cot\theta\frac{\partial}{\partial\phi}\right),$$

$$\xi_{15} + i\xi_{10} = \mathrm{e}^{-it}\left(-i\sin\theta\cos\phi\frac{\partial}{\partial t} + \cos\theta\cos\phi\frac{\partial}{\partial\theta} - \frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \right),$$

$$\xi_{25} + i\xi_{20} = \mathrm{e}^{-it}\left(-i\sin\theta\sin\phi\frac{\partial}{\partial t} + \cos\theta\sin\phi\frac{\partial}{\partial\theta} + \frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \right),$$

$$\xi_{35} + i\xi_{30} = \mathrm{e}^{-it}\left(-i\cos\theta\frac{\partial}{\partial t} - \sin\theta\frac{\partial}{\partial\theta} \right),$$

$$\Omega_{05} = 0, \qquad \Omega_{12} = 0, \qquad \Omega_{23} + i\Omega_{31} = 0,$$

$$\Omega_{15} + i\Omega_{10} = -\tfrac{1}{2}\mathrm{e}^{-it}\sin\theta\cos\phi,$$

$$\Omega_{25} + i\Omega_{20} = -\tfrac{1}{2}\mathrm{e}^{-it}\sin\theta\sin\phi,$$

$$\Omega_{35} + i\Omega_{30} = -\tfrac{1}{2}\mathrm{e}^{-it}\cos\theta.$$

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